Paraxial theory of slow self-focusing

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We present a theory of slow self-focusing that is paraxial in nature, gives the field including the phase and eikonal explicitly, while it also agrees with the results of variational and moments theories. After presenting the features of the theory, particularly its similarity to the central force problem, we go on to reformulate the theory for an absorbing medium. We find that the laser beam focuses to a constant beamwidth with a small phase-front curvature depending on the extent of absorption. The theory is applicable to a whole range of saturating nonlinearities although it specializes to two plasma cases, the ponderomotive force based and the relativistic electron quiver based nonlinearities, for definitive results.

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I. INTRODUCTION

With the importance of the phenomenon of self-focusing in the context of modern lasers and their applications [1,2], its theory continues to attract attention with newer more powerful methodologies being brought to bear on it. Unlike the ideal case of the two-dimensional self-focusing with cubic nonlinearity that found the classic solution in terms of the inverse-scattering techniques [3,7], self-focusing of cylindrical laser beams does not have such a unique elegant formulation. To mention a few of the recent theoretical attempts at self-focusing, we may broadly divide them into two categories: those theories that limit themselves to the cubic and quintic nonlinearities and those that consider the complete saturation of the nonlinearity and its consequences.

In the former category, the first of the recent attempts we mention is the work based on symmetry methods applied to the nonlinear Schrödinger equation, one due to Gagnon and Winternitz [4] and the other due to Clarkson and Hood [5]. Although these theories have not only been developed for cubic but also the quintic nonlinearity, no definite conclusions can be yet drawn about the physical nature of self-focusing in cylindrical geometry using these theories. Another independent method to tackle the problem of self-focusing in cylindrical geometry that holds a lot of promise in the future is that of Kovalev *et al.* [6] based on the renormalization group theory and has quite some success in this category of weak nonlinearities at least as much as the old paraxial theories of Akhmanov *et al.* [8] and the original moments method due to Vlasov *et al.* [9], both of which were developed for the cubic nonlinearity and generalized for the quintic nonlinearity. There is the problem of collapse of the cubic nonlinearity that is arrested by the quintic nonlinearity addressed in all the above papers but reaching maturity in the recent works of Luc Berge [10], Malkin [11], Sulem and Sulem [12], and Fibich and co-workers [13,14]. These theories include the cubic as well as the quintic nonlinearities and reach the interesting conclusion that such studies are enough to account for all saturating nonlinearities since anyway the final beam shape is that of a self-trapped solution of the cubic nonlinearity. Although there is some analytical tractability in these theories using the Talanov lens transformation [15] and the original moments theory of Vlasov *et al.* [9], there is no explicit tracking of the field form including the phase information like the development of the eikonal which is useful if self-focusing has to be studied in conjunction with other nonlinear phenomena of the laser beam.

Among the second category of theories that deal with saturating-type nonlinearities, the generalization of the Akhmanov *et al.* theory [8] by Sodha *et al.* [16,17] and Max [18] were the earliest. It was, however, shown by Lam *et al.* [19] that these generalizations of the Akhmanov *et al.* theories fail to agree with numerical results and went on to give a better theory for self-focusing in a saturating nonlinearity medium using generalization of the moments theory of Vlasov *et al.* [9]. Using another independent approach to self-focusing theory in saturating nonlinear medium, the variational method, this generalized theory of moments and the numerical results were vindicated by Anderson and co-workers [20,21]. The moments and the variational theories do not again give the explicit field form including the phase development in terms of the eikonal but assume a field intensity profile for obtaining the results. These theories are, therefore, unsuitable if other phase-dependent nonlinear processes need to be studied in conjunction with self-focusing. These two theories also have not been generalized for absorbing media. Many of these shortcomings have been overcome in a recent study that recovers the results of the moments and variational theories at least close to self-trapping using an explicit field form by using a proper paraxial field and dielectric constant expansion [22]. This theory has been studied for its results only partially [23,24] and has not been precisely compared with the moments and the variational theories of Lam *et al.* [19] and Anderson and Bonnedal [20] except for self-trapping.

Our concern in this paper is to develop a theory for cw (continuous-wave) self-focusing of laser beams for an arbitrary saturating nonlinearity for a slowly evolving regime based on the quasioptic or the paraxial approximation. The aim is to develop the theory *ab initio* in such a manner as to give exactly the same results as the moments and the variational theories of Lam *et al.* [19] and Anderson and Bonnedal [20], respectively and the corrected paraxial theory

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[22] in the absence of absorption of the medium. It goes beyond these theories, however, by including the explicit field form in terms of the eikonal and also generalizes to the case of absorbing medium, including both linear and nonlinear absorption. It also elaborates on the surprising similarity of the absorption less self-focusing problem to the central force problem.

In Sec. II we go on to set up the explicit field form starting from the paraxial quasioptic wave equation. The self-focusing dynamics in an absorptionless medium is discussed then in Sec. III including the similarities with the central force problem. We rederive the theory suitable to absorbing as well as nonabsorbing media in Sec. IV. The results of self-focusing in an absorbing medium are discussed in Sec. IV C. The limitations and future directions for developments of the theory are pointed out in Sec. V.

II. FIELD FORM FOR SLOW SELF-FOCUSING

With the original monochromatic wave electric field at frequency ω having dependence on space and time as

$$E=A(z,r)\exp[i(\omega t-kz)], \tag{1}$$

where $k^2=\epsilon_L\omega^2/c^2$ is the square of the linear wave number for the wave propagating in the z direction, the paraxial quasioptic scalar wave equation is [8,19,20]

$$-2ik\frac{\partial A}{\partial z}+\left(\frac{\partial^2 A}{\partial r^2}+\frac{1}{r}\frac{\partial A}{\partial r}\right)-\frac{k^2}{\epsilon_L}\Phi(AA^*)A=0, \tag{2}$$

where the intensity-dependent nonlinear dielectric constant

$$\epsilon(EE^*)=\epsilon_L-\Phi(EE^*) \tag{3}$$

has, respectively, the linear and nonlinear parts ϵ_L and $\Phi(EE^*)$, the latter of which saturates in general with the intensity factor ($EE^*=AA^*$) and is assumed to allow the Taylor-McLaurin expansion

$$\Phi(EE^*)=\sum_{n=1}^{\infty}a_n(EE^*)^n. \tag{4}$$

The field E and its amplitude A are normalized with respect to a suitable value of the variables derived for the particular nonlinearity (see Sec. II C). Here we will have $a_n=(\omega_p^2/\omega^2)\overline{a_n}$, a form suitable for plasma nonlinearities with the plasma frequency given by $\omega_p^2=4\pi n_0e^2/m$ (n_0 is the plasma number density and e and m are the electric charge and mass of the electron, respectively). Also the linear part of the dielectric constant will be $\epsilon_L=1-\omega_p^2/[\omega^2(1-i\nu/\omega)]$, ν being the effective collision frequency of the plasma. In an absorptionless plasma, $\nu=0$ as in this section.

The above generalized nonlinear Schrödinger equation will be used for analyzing self-focusing. Separating the beam envelope A_0 and the eikonal (phase) S through the substitution $A(r,z)=A_0(r,z)e^{-ikS(r,z)}$ and separation of real and imaginary parts gives the following two coupled evolution equations for the intensity and the eikonal of the electromagnetic wave in the nonlinear medium in the absence of absorption:

$$2\frac{\partial S}{\partial z}+\left(\frac{\partial S}{\partial r}\right)^2=-\frac{1}{\epsilon_L}\Phi(A_0A_0^*)+\frac{1}{k^2A_0}\left(\frac{\partial^2 A_0}{\partial r^2}+\frac{1}{r}\frac{\partial A_0}{\partial r}\right) \tag{5}$$

and

$$\frac{\partial A_0^2}{\partial z}+\frac{\partial S}{\partial r}\frac{\partial A_0^2}{\partial r}+A_0^2\left(\frac{\partial^2 S}{\partial r^2}+\frac{1}{r}\frac{\partial S}{\partial r}\right)=0. \tag{6}$$

The two terms on the right hand side of Eq. (5) determine the behavior of the eikonal S, i.e., the convergence or divergence of the beam, the first term determines nonlinear refraction while the second term determines diffraction. The second equation, Eq. (6), determines the evolution of the beam envelope. The method of analysis largely is made deliberately similar to the popular method by Akhmanov *et al.* [8] except for the manner in which $\Phi(EE^*)$ is approximated in the paraxial region that will be dealt with in Sec. II B below. We first outline for completeness sake the method of guessing out the envelope of the beam using Eq. (6).

A. Evolution of the beam profile

For a slightly converging/diverging beam with a spherical wave front of large radius of curvature, it is mathematically convenient to assume the following solution for the eikonal:

$$S=\frac{r^2}{2}\beta(z)+\varphi(z). \tag{7}$$

The surfaces of constant phase then correspond to surfaces $S(x,y)$ defined in Eq. (7) for each value of z. The Gaussian curvature of such a surface will be given by [28]

$$K=\frac{1}{R_xR_y}=\frac{S_{xx}S_{yy}-S_{xy}^2}{(1+S_x^2+S_y^2)^2}=\frac{\beta^2(z)}{(1+\beta^2r^2)^2}=\frac{1}{R^2},$$

where $r^2=x^2+y^2$ and the subscripts on S indicate partial differentiation; $R_x=R_y(=R)$ are the principal radii of curvature of the surface defined by $S(x,y)$ (for a given value of z). For $r\ll\beta^{-1}$, therefore $1/\beta(z)=R(z)$ represents the radius of curvature of the wave front. In addition, we introduce the real beam-width function $f(z)$ defined by the following equation:

$$\frac{1}{f}\frac{df}{dz}=\beta. \tag{8}$$

Equation (6) then takes the form

$$\frac{\partial A_0^2}{\partial z}+r\left(\frac{1}{f}\frac{df}{dz}\right)\frac{\partial A_0^2}{\partial r}+2\left(\frac{1}{f}\frac{df}{dz}\right)A_0^2=0. \tag{9}$$

Introducing a new variable $u=1/f$, we can write

$$-u\frac{\partial A_0^2}{\partial u}+r\frac{\partial A_0^2}{\partial r}+2A_0^2=0.$$

Further, introducing two more independent variables $\xi=(r/r_0)u$, and $\chi=r/(r_0u)$, where r_0 is the scale length in the radial direction for the laser beam, the above equation can be written in the form [8]

$$\chi\frac{\partial A_0^2}{\partial \chi}+A_0^2=0$$

or

$$\frac{\partial}{\partial \chi}(\chi A_0^2)=0$$

or χA_0^2 is equal to an arbitrary function of ξ. Since $\chi=r/(r_0u)=\xi/u^2$, one concludes that $A_0^2=u^2\mathsf{F}(\xi)$ in general. Hence we find that the general solution of Eq. (2) is a self-similar form of the intensity given as

$$A_0^2=\frac{E_0^2}{f^2}\,\mathsf{F}\left(\frac{r}{r_0f(z)}\right), \tag{10}$$

where F is an arbitrary function of its argument. $f(z)$ turns out from above to be determining two aspects of the beam envelope. It represents the beamwidth on multiplication with r_0 at any arbitrary value of z. f also determines the amplitude of the electromagnetic beam in the form E_0/f at any arbitrary distance z during propagation and focusing. Equation (8) may also be considered a result of the conservation of the photon number $N=\int AA^*2\pi r dr$, which is a known invariant of Eq. (2).

The above equation clearly indicates that the intensity on the axis, i.e., at $r=0$, increases as f decreases; hence the minimum value of f^2 corresponds to the focus where the intensity is maximum and the beamwidth is a minimum. Writing out $A_0=(E_0/f)\mathsf{F}^{1/2}(\xi)$, from Eq. (10), we can now write Eq. (5) in the following form ($\eta=z/kr_0^2$):

$$(\mathsf{F}^{1/2})''+\frac{1}{\xi}(F^{1/2})' =\left[\xi^2f^3\frac{d^2f}{d\eta^2}+f^2k_0^2r_0^2\Phi\left(\frac{E_0^2}{f^2}\,\mathsf{F}\right)+2f^2\frac{d\phi}{d\eta}\right]\mathsf{F}^{1/2}, \tag{11}$$

where $\phi=k\varphi$. This is the equation to be solved in general for $\mathsf{F}^{1/2}$.

It is very suggestive in view of the expansions of the form in Eq. (16) that are to follow, that a special case of interest (but not the only choice allowed by this theory for a laser beam in a nonlinear self-focusing medium; see Eq. (74) for elaboration) is when the initial intensity distribution (along the radius) may be assumed to be a Gaussian

$$A_0^2(r,z=0)=E_0^2\exp\left[\frac{-r^2}{r_0^2}\right] \tag{12}$$

as if the laser is introduced in the TEM_{00} mode into the self-focusing medium. The intensity distribution using Eq. (10) reduces to the self-similar form

$$A_0^2(r,z)=\frac{E_0^2}{f^2(z)}\exp\left[\frac{-r^2}{r_0^2f^2(z)}\right] \tag{13}$$

or

$$A_0=\frac{E_0}{f(z)}\exp\left[\frac{-r^2}{2r_0^2f^2(z)}\right]. \tag{14}$$

B. Beamwidth equation

Following Refs. [24,25] we next use the corrected power series expansion which is not a Taylor series expansion in configuration space for $\Phi(EE^*)$ or $\epsilon(EE^*)$ (but obtained using Taylor expansion in a suitable momentum space) that may be written as

$$\epsilon(EE^*)\simeq\epsilon_0-\epsilon_2r^2, \tag{15}$$

which gives the following expression for $\Phi(EE^*)$:

$$\Phi(EE^*)=(\epsilon_L-\epsilon_0)+\epsilon_2r^2, \tag{16}$$

where [24,25]

$$\epsilon_0=\epsilon_L-\sum_{n=1}^{\infty}a_n\frac{(2n+1)}{(n+1)^2}\left(\frac{E_0^2}{f^2}\right)^n, \tag{17}$$

$$\epsilon_2=\frac{-1}{r_0^2f^2}\sum_{n=1}^{\infty}a_n\frac{n}{(n+1)^2}\left(\frac{E_0^2}{f^2}\right)^n \tag{18}$$

specializing to the case of the Gaussian beam. Equation (5) or (11) then simplifies to the following equation by equating the coefficient of r^2 on both sides:

$$\frac{d^2f}{d\eta^2}=-\frac{\epsilon_2k^2r_0^4}{\epsilon_L}f+\frac{1}{f^3}=F(f), \tag{19}$$

where $\eta=z/kr_0^2$, the propagation distance measured in terms of the Rayleigh length kr_0^2. This beamwidth equation along with the field intensity from Eq. (13) is exactly the same as obtained in the moments and variational theories [19–21] and will be used for further computations in the following section.

The determining equation for $\varphi(z)$ can be obtained from Eq. (6) or (11) on equating the r^2-independent terms and gives [26]

$$2\frac{d\phi}{dz}=k\left(\frac{\epsilon_0-\epsilon_L}{\epsilon_L}\right)-\frac{2}{kr_0^2f^2}, \tag{20}$$

where $\phi=k\varphi$. For self-focusing the boundary condition at $z=0$, $f=1$, and $df/dz=0$ will often be used. The latter condition implies that the beam is a plane wave at $z=0$ while the former condition defines r_0 as the beamwidth at $z=0$.

In the expressions for ϵ_0 and ϵ_2 in Eqs. (17) and (18) above, the summations over the power series are related to the power series of Φ given in Eq. (4) and are convenient for the saturating nonlinearity related calculations for small values of the normalized intensity. For a more general expression valid for all ranges of the normalized intensity, it is convenient to sum up the series at least in an integral form if not in a closed form. The following series and their integral representations will be involved in the evaluation of ϵ_0 and ϵ_2. The first sum of interest is (with $x=EE^*$)

$$S(x)=\sum_{n=1}^{\infty}\frac{a_n}{(n+1)^2}x^n, \tag{21}$$

where the summation in the integral form can be written implying term-by-term integration as follows

$$S(x)=\frac{1}{x}\int_0^x\frac{1}{x'}\left[\int_0^{x'}\Phi(x'')dx''\right]dx' \tag{22}$$

with Φ being defined from Eq. (4) for the particular nonlinearity of interest. If we define $\tilde{S}(x)=\int_0^x\Phi(x)dx$, we have $S(x)=1/x\int_0^x 1/x\tilde{S}(x)dx$. It can be verified for all plasma nonlinearities for which $a_n=\omega_p^2/\omega^2\overline{a_n}$ that $S(x)\to 0$ as $x\to 0$ and $S(x)\to-\omega_p^2/\omega^2$ as $x\to\infty$ as expected always. Another series of interest that can be evaluated using $S(x)$ is $S_2(x)=\Sigma_{n=1}^{\infty}a_n n/(n+1)^2x^n=x\,dS(x)/dx$. Specific cases of evaluation are given below [27].

C. Two specific plasma nonlinearities

The explicit calculations in this paper are performed for two nonlinearities of interest in plasma physics although the theory remains valid for any steady state nonlinearity that has a relaxation time much smaller than the laser pulse length. Since anyway the theory is for cw laser beams, this requirement is easily met.

1. The ponderomotive nonlinearity

The first of the considered nonlinearities is that which arises out of the ponderomotive force of the laser radiation [29] that expels the electrons from the strong field regions in the transverse direction thereby creating a low density or high refractive index channel for the laser in the plasma. The ponderomotive nonlinearity sets in with a relaxation time $\tau_p\sim(r_o/v_a)\sim 10^{-9}$–$10^{-11}$ sec; here r_o is the transverse beam inhomogeneity scale length and v_a is the speed of the ion acoustic wave. Short high-power laser pulses in the nanosecond to picosecond range are sensitive to the ponderomotive mechanism. When the laser pulse width is much large compared to τ_p, the resultant dielectric constant of the plasma will be [29]

$$\epsilon=1-\left(\frac{\omega_p}{\omega}\right)^2\exp(-EE^*/E_{00}^2) \tag{23}$$

where $E_{00}^2=8m\omega^2k_BT/e^2$, k_B is the Boltzmann constant, T is the common temperature of the electrons and the ions, e and m are the charge and mass of an electron, M is the mass of an ion, $\omega_p=\sqrt{(}4\pi n_0e^2/m$ is the plasma frequency in the absence of EM wave where n_0 is the number density of the plasma and ω is the frequency of the electromagnetic (EM) wave. Throughout this paper, the normalized field intensity factor is simply written as EE^*, it being understood that it actually stands for EE^*/E_{00}^2, and the normalizing field square in the denominator has been absorbed into the new notation.

For this nonlinearity, the expression for $\Phi(EE^*)$ of Eq. (4) will be

$$\Phi(EE^*)=\frac{\omega_p^2}{\omega^2}(e^{(-EE^*)}-1)=\frac{\omega_p^2}{\omega^2}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}(EE^*)^n\right)$$

so that for the ponderomotive nonlinearity, $\overline{a_n}=(-1)^n/n!$ and $a_n=(\omega_p^2/\omega^2)\overline{a_n}$.

We may also write for the case of ponderomotive nonlinearity with some effort using Eq. (22), $S(x)=(\omega_p^2/\omega^2)\overline{S}(x)$

$$\overline{S}(x)=\frac{-1}{x}[-E_1(x)-\gamma-\ln(x)+x]=\left[-1+\frac{E(x)}{x}\right], \tag{24}$$

where the exponential integral [31] $E_1(x)=\int_x^{\infty}(e^{-x}/x)dx$; $\gamma=0.577\ldots$, Euler's constant and $E(x)=\int_0^x[(1-e^{-t})/t]dt=E_1(x)+\gamma+\ln(x)$. We find $S(x)\to 0$ as $x\to 0$ and $S(x)\to-1/x[-E_1(x)-\gamma-\ln(x)+x]\to-1$ as $x\to\infty$.

One also can easily evaluate $\overline{S_2}=x\,d\overline{S}/dx$ explicitly as

$$\overline{S_2}(x)=\left[\frac{(1-e^{-x})}{x}-\frac{E(x)}{x}\right]. \tag{25}$$

One may verify that $\overline{S_2}\to 0$ as $x\to 0$ as well as when $x\to\infty$.

2. The relativistic nonlinearity

The relativistic nonlinearity arises because of the intense quiver of the electron in the field of the strong laser electromagnetic field [30]. The normalization field for the relativistic nonlinearity, $E_{00}^2=4m_0^2\omega^2c^2/e^2$, is much higher compared to the ponderomotive nonlinearity. The nonlinearity gets established instantaneously so that it is essentially governed by the rise time of the laser pulse for its manifestation. For very short laser pulses such as the femtosecond pulses, only the relativistic nonlinearity dominates. We assume such a situation where the ponderomotive and relativistic nonlinearities dominate in different regimes.

For relativistic nonlinearity in a plasma the dielectric constant is [30]

$$\epsilon(EE^*)=1-\frac{\omega_p^2}{\omega^2}\frac{1}{(1+EE^*)^{1/2}}=\epsilon_L-\Phi(EE^*), \tag{26}$$

where $\epsilon_L=1-\omega_p^2/\omega^2$ again and

$$\Phi(EE^*)=\frac{\omega_p^2}{\omega^2}\left[\frac{1}{(1+EE^*)^{1/2}}-1\right]=\frac{\omega_p^2}{\omega^2}\left[\sum_{n=1}^{\infty}\bar{a}_n(EE^*)^n\right], \tag{27}$$

where

$$\bar{a}_r=(-1)^r\frac{1\times3\times5\times\ldots\times(2r-1)}{2^r r!}. \tag{28}$$

Note that the field intensity factor EE^* has again to be reread as EE^*/E_{00}^2 with the appropriate value of E_{00}^2 for the relativistic nonlinearity quoted above. In this case the expressions for $S(x)$ and $S_2(x)$ can be worked out to be the following, again using Eqs. (21) and (22):

$$S(x)=\frac{\omega_p^2}{\omega^2}\left[4\left\{\left(\frac{\sqrt{1+x}-1}{x}-\frac{1}{x}\ln\left(\frac{1}{2}\right)(1+\sqrt{1+x}\right\}-1\right]. \tag{29}$$

It can be verified that in this case also, $S(x)\to0$ as $x\to0$ and $S(x)\to-\omega_p^2/\omega^2$ as $x\to\infty$ as expected always. Also the series $S_2(x)=\Sigma_{n=1}^{\infty}a_n(n/(n+1)^2)x^n=xdS(x)/dx$ can now be evaluated

$$\bar{S}_2(x)=\left[-[\bar{S}(x)+1]+\frac{2}{1+\sqrt{1+x}}\right], \tag{30}$$

where $\bar{S}(x)=S(x)\omega_p^2/\omega^2, S_2(x)=\bar{S}_2(x)\omega_p^2/\omega^2$. We find again for this nonlinearity also, $S_2(x)\to0$ as $x\to0$ and $S_2(x)\to0$ also as $x\to\infty$.

III. SELF-FOCUSING DYNAMICS IN A NONABSORBING MEDIUM

There is much to learn from the absorptionless case of slow self-focusing. We present these interesting results in this section using Eqs. (19) and (20). Before going over to the more complicated self-focusing dynamics, we first give the results of the special case of self-trapping of the laser beam that represents the important result of the beam diffraction exactly balancing out the nonlinear focusing of the beam.

A. Self-trapping

We ask the question whether a Gaussian beam with initial radius ($r_0^2/2$) with a plane wave front (implying $(df/d\eta)_{z=0}=0$) introduced into the self-focusing medium at $z=0$ [as in Eq. (2)] can continue to propagate in a diffractionless manner by balancing the diffraction with nonlinear refraction, so that $f=1$ for all z in Eq. (14). The question gets answered by substituting this constant value for f in Eq. (19) which immediately implies the right hand side vanishes also as the left hand side, giving us the condition

$$\frac{r_0\omega_p}{c}=\left(\frac{-1}{\bar{S}_2(x)}\right)^{1/2} \tag{31}$$

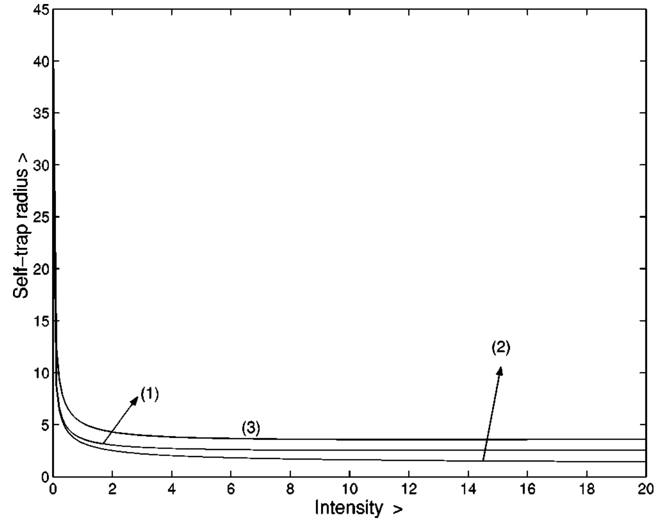

FIG. 1. Normalized self-trapped radius $\omega_p r_0/c$ plotted against normalized intensity $I=E_0^2/E_{00}^2$ for normal self-trapping based on Eq. (31) (curve 1) and for self-trapping of the second kind based on Eq. (37)(curve 2) for relativistic nonlinearity (as in Sec. II C 2). The exact numerical solution of Eq. (2) for self-trapping yields curve 3.

for self-trapping. Substituting for $S_2(x)$ for $x=E_0^2$ for the specific nonlinearity either from Eq. (25) for the ponderomotive nonlinearity or Eq. (30) for the relativistic nonlinearity yields the condition for self-trapping explicitly. In Fig. 1 we have given the result for the relativistic nonlinearity. Also for comparison is given the exact solution by solving Eq. (2) directly numerically for the self-trapping case. The agreement is reasonable as can be seen.

For more discussion on self-trapping, particularly, the self-trapping of the second kind and most stable self-trapping values at f^*, see the following section.

B. The potential function for self-focusing dynamics

It is possible in the absorptionless case when $\epsilon_2(EE^*)$ is real to write out a first integral of Eq. (19) in the form of a Hamiltonian or energy principle that will serve as a key to the analysis and computations. Multiplying Eq. (19) by $df/d\eta$ and integrating, we get the equation

$$\frac{1}{2}\left(\frac{df}{d\eta}\right)^2+U(f)=E, \tag{32}$$

where E is the constant of integration, the Hamiltonian, or energy function and $U(f)=-\int^f F(f)df$. More explicitly, we get

$$U(f)=-\int^f\left[\frac{1}{f}\frac{\omega_p^2r_0^2}{c^2}\sum_{n=1}^{\infty}\bar{a}_n\frac{n}{(n+1)^2}\left(\frac{E_0^2}{f^2}\right)^n+\frac{1}{f^3}\right]df \tag{33}$$

$$=\frac{1}{2}\frac{\omega_p^2r_0^2}{c^2}\sum_{n=1}^{\infty}\bar{a}_n\frac{1}{(n+1)^2}\left(\frac{E_0^2}{f^2}\right)^n+\frac{1}{2f^2} \tag{34}$$

or

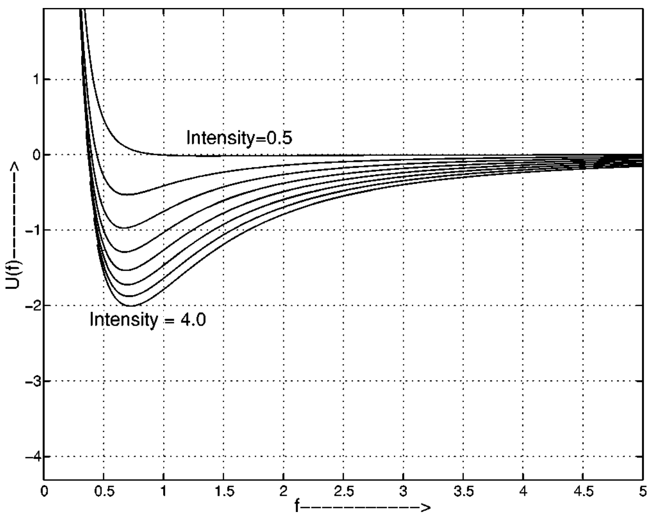

FIG. 2. Variation of the potential function for self-focusing, $U(f)$, with f in the case of ponderomotive nonlinearity for various values of normalized intensity $I = E_0^2/E_{00}^2$ and normalized self-trapping radius $\omega_p r_0/c = 3.0$ from Eq. (35). The well in the potential disappears for low intensities for which self-focusing is not possible.

$$U(f) = \frac{1}{2f^2} + \frac{1}{2}\frac{\omega_p^2 r_0^2}{c^2}\overline{S}(E_0^2/f^2). \tag{35}$$

We can easily find that $U(f) \to \infty$ as $f \to 0$ (because of the domination of the first term that corresponds to diffractive tendency of the beam) and $U(f) \to 0$ as $f \to \infty$ because both terms on the right hand side tend to zero in this limit. (Note that the limit of this potential function evaluated by Anderson and Bonnedal [20] as $f \to \infty$ was incorrect and could lead to incorrect conclusions.) Figure 2 illustrates the behavior of $U(f)$ for various values of the parameters, E_0^2, and for $\overline{r_0} = \omega_p r_0/c = 3.0$ in the case of the ponderomotive nonlinearity for which $S(x)$ is used from Eq. (24). The change of sign of $U(f)$ from positive to negative values goes on to show that a potential well occurs only above the self-focusing threshold, i.e., when the parameter values of E_0^2 and $\overline{r_0} = \omega_p r_0/c$ are above the self-trapping threshold curve as in Fig. 1.

Writing out the kinetic energy and evaluating the velocity

$$\frac{df}{d\eta} = \pm\sqrt{2[E - U(f)]}, \tag{36}$$

one observes that there are three possibilities depending on the values of the total energy E. For $E > 0$, the line $E = \text{const}$ intersects the $U(f)$ curve for positive values of the potential at a single point and the velocity could be real but it implies defocusing since in this region the diffraction term in $1/f^2$ dominates the focusing term. The beam can focus till $E = \text{const}$ meets the $U(f)$ curve and then it goes on to defocus indefinitely. In the potential well where $U(f) < 0$, one needs $E < 0$ values for intersection of the $E = \text{const}$ curves with the $U(f)$ curves. The intersection takes place at two points say at, f_m and $f_M > f_m$, and the laser beam is constrained to focus and defocus repeatedly between these two limits of f. At the bottom of the potential well, there is again only one point of contact pointing to the possibility of the beam being able to maintain a fixed value of f there. Indeed, this is the self-trapping situation where $f = \text{const}$ can be maintained and it is a stable fixed point of the system since it is at the bottom of the potential well.

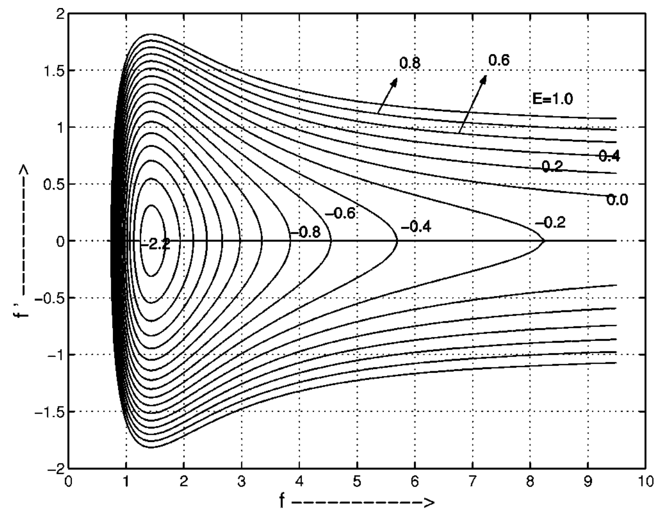

FIG. 3. Contour plots of constant energy E from Eq. (36) generated for the bottom curve ($E_0^2/E_{00}^2 = 4$, $\omega_p r_0/c = 3$) of the potential well in Fig. 2 for ponderomotive nonlinearity in the ($f, f' = df/d\eta$) plane.

Varying the values of E and plotting $df/d\eta$ with f we get the self-trapping contour plots of Fig. 3. We see from these curves as we move to the minimum of potential well, $U(f)$, where the well is more symmetric, that the curves are closed curves and more circular. This minimum of the potential well corresponds to the point where the beam is stable towards perturbations. For these curves, we fix the value of intensity and the self-trapping radius. One may write Eq. (19) in the form

$$\frac{d^2 f}{d\eta^2} = -\frac{dU}{df} = F(f).$$

At the minimum of $U(f)$ we have $dU/df = 0$ implying that at these points $F(f) = 0$, which goes on to define the fixed point of the system. The fixed point defines the most stable of the values of $f(=f^*)$ for which self-trapping is possible. Correspondingly, the value $(r_0 f^*)$ will be the value of the beamwidth for which the system is stable. Figure 4 depicts the values of f^* for some value of $\overline{r_0}$. In Sec. III C 1 [Eq. (55) and the discussion following it] we return to show that the beam executes simple-harmonic motion around these values of f^* thereby showing stability under small perturbation.

The value of $f = 1$ (as good as any other value of $f = \text{const}$) of the preceding section is one of the values for which self-trapping can occur although it is not the most stable value. The value of f^* differs from $f = 1$ in general and hence one concludes that if the beam is given a chance somehow to evolve, it will rather evolve towards the value f^*. This chance to evolve actually is realized in the presence of at least some effective absorption in the medium as dis-

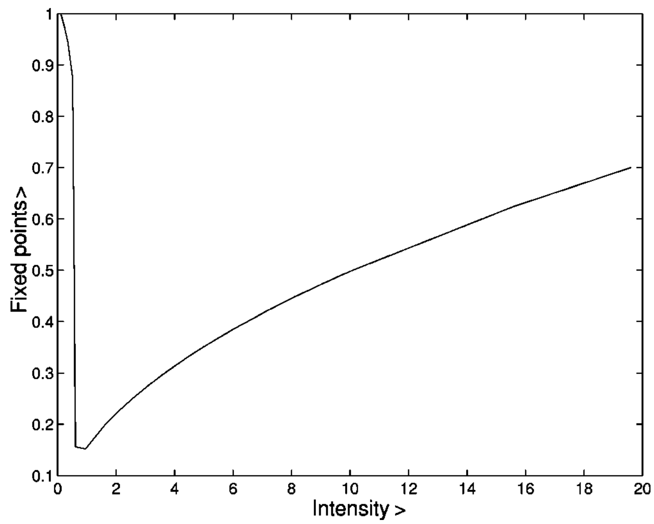

FIG. 4. Fixed point f^* with intensity E_0^2/E_{00}^2 ($\omega_p r_0/c=3.0$)

cussed in Sec. IV C. Note that the boundary conditions at $\eta=0$, $f=1$ and $df/d\eta=0$ give us $U(1)=E$ in Eq. (36), and these values of E for given parameters $E_0^2,\overline{r}_0$ may easily be estimated.

In Fig. 3 we give, corresponding to the ponderomotive potential of Fig. 2, the contours of velocity variable $df/d\eta$ on a phase-space plot with respect to values of f. We can see that for $E>0$ the beam defocuses even if it started off focusing. For $E<0$ the beam oscillates within the well to give closed curves in phase space. The eye of the closed contours corresponds to the bottom of the potential well that is the fixed point of the system for the given system parameters $E_0^2,\overline{r}_0$.

A very special value of E in Fig. 3 is $E=0$, the one on the separatrix that separates the closed contours (for which $E<0$) from the open ended contours (for which $E>0$) in the phase plane. On this contour the beam takes infinite distance to travel, first focusing, if the boundary condition is so chosen, and finally defocusing again in infinite distance. Effectively, the beam travels unperturbed, i.e., it travels with diffraction exactly canceling nonlinear focusing. The condition for this diffractionless propagation (i.e., for $f=1$) *or self-trapping of the second kind* is, from Eqs. (32) and (35),

$$\left(1-\frac{1}{\overline{R}^2}\right)\frac{1}{\overline{r}_0^2}=-\overline{S}(E_0^2),$$

where Eqs. (11) and (12) have been used to express the radius of curvature R of the wave front and its dimensionless form $\overline{R}=R/kr_0^2$. In the special circumstance that the wave has a plane wave front implying $\overline{R}\rightarrow\infty$, the condition for this self-trapping of the second kind reads

$$\overline{r}_0=\left(\frac{-1}{\overline{S}(E_0^2)}\right)^{1/2} \tag{37}$$

This condition has also been plotted in Fig. 1 to show a comparison with the condition for self-trapping of the first kind that comes from Eq. (31). For finite values of $\overline{R}$, this self-trapping curve lies below the one shown in Fig. 1 but in the first quadrant. Note that this self-trapping of the second kind is unstable in the sense that deviation from the exact self-trapping conditions could lead to either indefinite defocusing or periodic focusing in the saturating nonlinearity medium. This is unlike the self-trapping of the first kind that is stable in general and most stable for values of f^* discussed above.

The dynamics of self-focusing are often of interest with the boundary condition $f=1$ and $df/d\eta=0$ at $\eta=z=0$. These plots show oscillations above the threshold of the first kind or the second kind of self-trapping of Fig. 1 and are depicted in Fig. 5 as phase-space plots. With distance this implies periodic focusing or defocusing.

C. Similarity of self-focusing dynamics to the central force problem

In the above section, the potential function just touches on some similarity of the absorptionless slow self-focusing dynamics problem in the (corrected) paraxial regime with the central force problem with an attractive potential. There are surprisingly many more similarities of the potential function $U(f)$ to that of Kepler or rather the attractive central force problem—too close to be ignored. One similarity is, of course, that the potential $U(f)$ is dependent on the radial coordinate as $U(r)\rightarrow\infty(r=f)$ as r or $f\rightarrow 0$ because of centrifugal effective potential and $U(r)$ or $U(f)\rightarrow 0$ as r or $f\rightarrow\infty$ because of the nature of the central potential.

First, the beamwidth parameter f may be identified as the radial coordinate in polar coordinate system. Then one can also "invent" a polar angle coordinate θ_f so that (f,θ_f) define a plane in the polar coordinate system. In a corresponding Cartesian system of coordinate we introduce the coordinates (f_1,f_2) such that

$$f=\sqrt{f_1^2+f_2^2},\quad \theta_f=\tan^{-1}\frac{f_2}{f_1}, \tag{38}$$

implying, $f_1=f\cos(\theta_f)$, $f_2=f\sin(\theta_f)$. The Lagrangian for a particle of unit mass may now be defined in the $(f,\ \theta_f)$ plane as the difference between the kinetic energy T and the potential energy $V(f)$, $L=T-V$ or explicitly

$$L=\frac{1}{2}\left[\left(\frac{df}{d\eta}\right)^2+f^2\left(\frac{d\theta_f}{d\eta}\right)^2\right]-V(f)=\tfrac{1}{2}[\dot{f}^2+f^2\dot{\theta}_f{}^2]-V(f).$$

It follows that the canonical momentum coordinate associated with the angle coordinate θ_f is

$$p_{\theta_f}=\frac{dL}{d\dot{\theta}_f}=f^2\dot{\theta}_f. \tag{39}$$

The corresponding Lagrange equation is

$$\dot{p}_{\theta_f}=\frac{d}{d\eta}(f^2\dot{\theta}_f)=\frac{\partial L}{\partial\theta_f}=0. \tag{40}$$

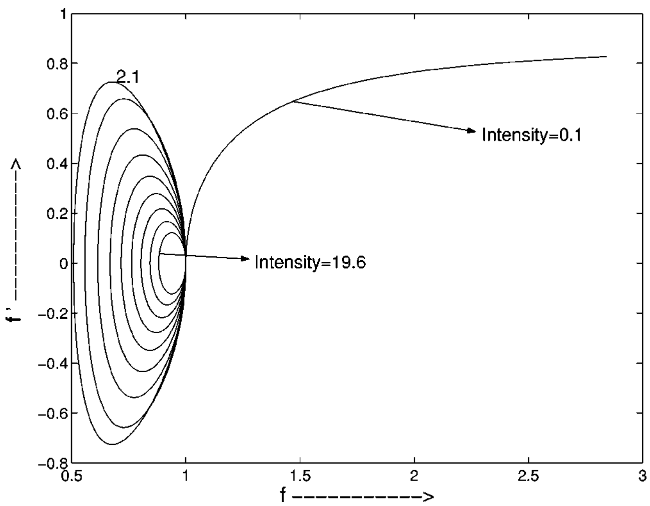

FIG. 5. Self-focusing phase-space plot in the $(f, f' = df/d\eta)$ plane for $\omega_p r_0/c = 3.0$ for various intensities. The closed curves indicate oscillations of f and $df/d\eta$ with η.

Hence, θ_f is a cyclic coordinate and

$$p_{\theta_f} = f^2 \dot{\theta}_f = l = (\text{constant in}\, \eta). \tag{41}$$

This implies conservation of angular momentum in the (f, θ_f) plane. One consequence is that we can always write

$$\dot{\theta}_f = \frac{l}{f^2}. \tag{42}$$

A suitable value of l, the constant angular momentum, is to be determined later.

We may define a three-dimensional space (f, θ_f, ψ_f) and define in turn an angular momentum vector **L** and because of the nature of the central potential that does not depend on θ_f and ψ_f, we may claim that the η derivative of **L**, the torque, vanishes keeping the vector **L** constant. The constancy in ψ_f direction of **L** ensures that the motion is restricted to the $(f,\ \theta_f)$ plane that is defined for, say, $\psi_f = \pi/2 = \text{const}$ and the magnitude of $\mathbf{L} = l$ constancy has just been shown to imply a relation between $\dot{\theta}_f$ and f^2. The proof of the constancy of **L** will be similar to that given in standard classical mechanics books e.g., Goldstein [32].

One may proceed also in the usual manner to define an infinitesimal area as $dA = \frac{1}{2} f(f\, d\theta_f)$ and an areal velocity as

$$\frac{dA}{d\eta} = \frac{1}{2} f^2 \frac{d\theta_f}{d\eta} = \frac{1}{2} f^2 \dot{\theta}_f = \frac{1}{2} l. \tag{43}$$

One then concludes that the constancy of the angular momentum l implies the constancy of the areal velocity in the slow paraxial self-focusing problem in the orbit produced in the $(f,\ \theta_f)$ plane. The main Lagrangian equation is the radial equation for f and its η derivatives that can now be written down as

$$\frac{d}{d\eta}\dot{f} - f \dot{\theta}_f{}^2 = -\frac{\partial V}{\partial f} = F(f),$$

where we call $F(f)$ the force in the f space just as $V(f)$ is the potential. Using the constancy of the angular momentum this can be rewritten as

$$\frac{d^2 f}{d\eta^2} = F(f) + \frac{l^2}{f^3} = -\frac{d}{df}\left[V(f) + \frac{l^2}{2f^2} \right].$$

The first integral of this after multiplying with $df/d\eta$ gives

$$\frac{1}{2}\left(\dot{f}^2 + \frac{l^2}{f^2} \right) + V(f) = E = \ \text{const.}$$

This may be recognized as the energy integral particularly after rewriting $l^2/f^2 = f^2 \dot{\theta}_f{}^2$. Writing the effective potential

$$U(f) = V(f) + \frac{l^2}{2f^2} \tag{44}$$

and on comparison with Eqs. (32) and (35), we may identify our $V(f)$ as

$$V(f) = \frac{1}{2}\overline{r_0^2}\ \overline{S}\left(\frac{E_0^2}{f^2} \right) \tag{45}$$

and

$$l = 1. \tag{46}$$

The latter result is particularly interesting and curious since *it identifies the origin of the* $1/f^2$ *term, the well-known Gaussian laser-beam diffraction term, as arising on account of a certain angular momentum conservation in the* $(f,\ \theta_f)$ *plane*.

In the $(f,\ \theta_f)$ plane, the orbit equation for $f(\theta_f)$ is another description of self-focusing. This is obtained by noting that the angular momentum constancy gives $d\eta = (f^2/l)\, d\theta_f$ and hence

$$\frac{d^2 f}{d\eta^2} = \frac{l^2}{f^2} \frac{d}{d\theta_f} \frac{1}{f^2} \frac{df}{d\theta_f} = -l^2 u^2 \frac{d^2 u}{d\theta_f^2}, \tag{47}$$

where $u = 1/f$ so that the earlier force equation can now be written as

$$\frac{d^2 u}{d\theta_f^2} + u = -\frac{1}{l^2 u^2} \tilde{F}\left(\frac{1}{u} \right), \tag{48}$$

where $\tilde{F}(1/u) = F(f)$. This differential equation for $u(\theta_f)$ or $f(\theta_f)$ can be integrated to give the orbit for self-focusing in the $(f,\ \theta_f)$ plane. In our case

$$\tilde{F}\left(\frac{1}{u} \right) = F(f) = -\frac{\partial V}{\partial f} = \frac{1}{2}\, \overline{r_0^2} u^2 \frac{d}{du}\, \overline{S}(E_0^2 u^2). \tag{49}$$

An illustrative orbit is given in Fig. 6. We find that as we increase the η values, the orbits remain between a minimum and a maximum value of f. This implies that the orbit is bounded but ergodic (see discussion at the end of Sec. III C 1 below).

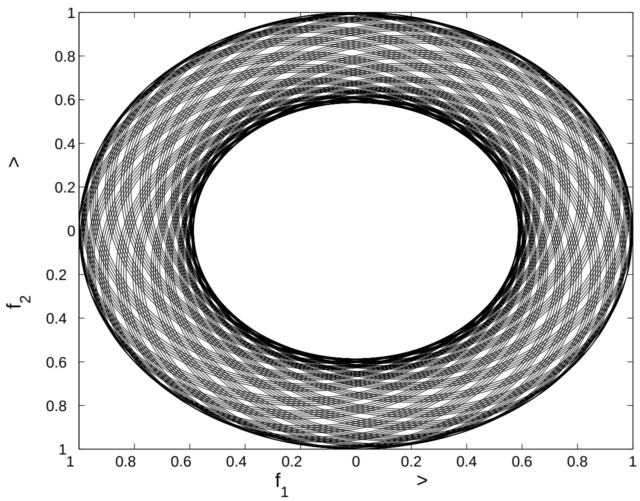

FIG. 6. The self-focusing orbit in the (f_1, f_2) plane. $\omega_p r_0/c = 5.0$, $E_0^2/E_{00}^2 = 3.0$, $\eta = 0$ to 140.0. The orbit is ergodic and space filling in the region $f_m < f < f_M$.

Note that $df/d\theta_f = (1/l) f^2 (df/d\eta)$. Hence the phase plot $(f, df/d\eta)$ can easily be converted to these polar plots through the relation $d\theta_f = (l/f^2) d\eta$. One may calculate θ_f in this manner and increment to obtain the orbit in the (f_1, f_2) plane. The boundary conditions at $\eta = 0$ are $f = 1$ $(u = 1)$ and $du/d\eta = df/d\eta = 0$ or $du/d\theta_f = 0$.

1. Periodicity or focal length for self-focusing for saturating nonlinearities

The orbit for negative energies $E = -|E|$ has two turning points in f between its maximum value f_M and minimum value f_m. Also the energy equation, Eq. (36), defines the radial momentum as

$$p = \frac{df}{d\eta} = \pm \sqrt{\left[2E - \frac{1}{f^2} - \overline{r_0^2 S\left(\frac{E_0^2}{f^2}\right)}\right]}. \tag{50}$$

The action variable is

$$J = \oint p\,df = \oint \sqrt{\left(2E - \frac{1}{f^2} - \frac{\omega_p^2 r_0^2}{c^2}\overline{S(f)}\right)}\,df. \tag{51}$$

The best manner to evaluate the integral is a numerical scheme that recognizes that (action) $= J =$ (area in the phase plane *or*

$$J = \oint \frac{df}{d\eta} df = 2\int_{f_m}^{f_M} \frac{df}{d\eta} df$$
$$= 2\int_{f_m}^{f_M} \sqrt{2[E - U(f)]}\,df.$$

The above integral was evaluated numerically to calculate the values of J. Values of E on the phase-space orbit are those obtained in Fig. 3. It can be seen from E vs J graphs in Fig. 7 that as the magnitude of E increases or as we move to the minimum of the potential $U(f)$, the area, J reduces. For

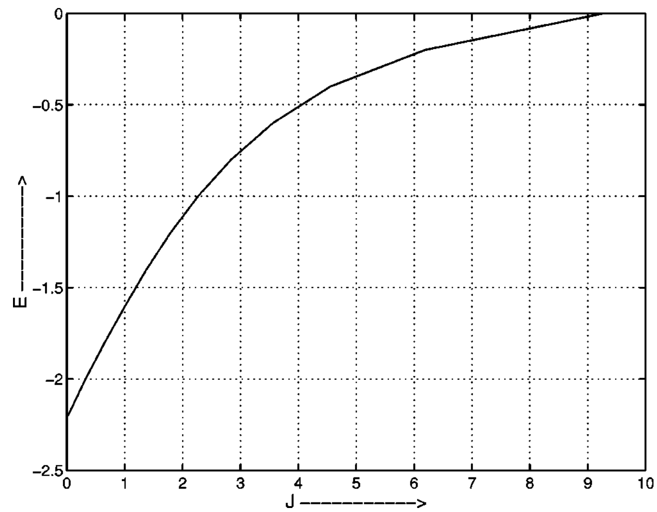

FIG. 7. Energy E plotted against the radial action J for $\omega_p r_0/c = 5.0$; $E_0^2/E_{00}^2 = 3.0$.

values of $E > 0$ action J becomes infinite. This corresponds to open contours in the phase plots.

Spatial frequency of periodic focusing, ν, is given as

$$\nu = \frac{dE}{dJ} \cong \frac{\Delta E}{\Delta J}, \tag{52}$$

The spatial period can be evaluated from here as

$$2\eta_f = \frac{1}{\nu} \Rightarrow \eta_f = \frac{1}{2\nu}, \tag{53}$$

where η_f is the dimensionless focal length or "period length" for the self-focusing process in the saturating medium. Using the numerical values obtained for E and J in the above graphs we get the variation of focal length z_f with E for both ponderomotive and relativistic nonlinearities, the former of which is depicted in Fig. 8.

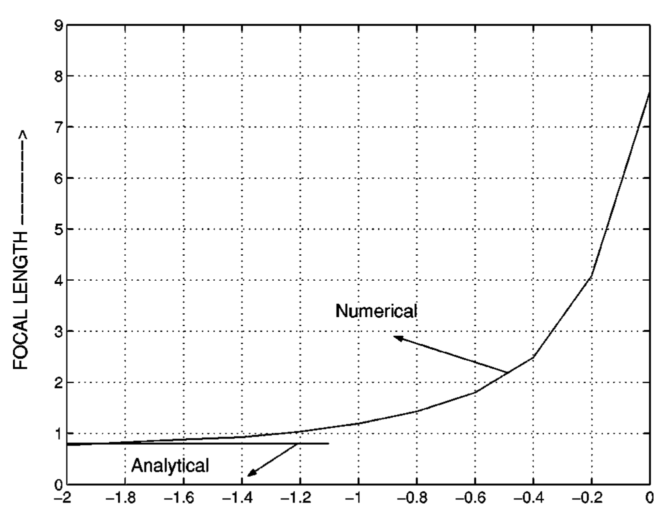

FIG. 8. Periodicity length η_f calculated using Eq. (53) and Fig. 7 for $\omega_p r_0/c = 5.0$; $E_0^2/E_{00}^2 = 3.0$. Analytic result is from Eq. (55).

Focal length could also be plotted as a function of intensity E_0^2 using the following expression:

$$E=\frac{1}{2}\left(\frac{df}{d\eta}\right)^2_{\eta=0}+\frac{\overline{r_0^2}}{2}\,\overline{S}(E_0^2)+\frac{1}{2}. \tag{54}$$

Analytical evaluation of the action J is straightforward at the bottom of the potential well. We may write near this point, $E=U(f^*)+\delta E$ or $\delta E=E-U(f^*)=|U(f^*)|-|E|$ so that [here $\delta f=(f-f^*)$],

$$2U(f)=2U(f^*)+\left.\frac{d^2U}{df^2}\right|_{f^*}\delta f^2,$$

where the first derivative is absent because the point where the power series expansion is being carried out, viz., $f=f^*$, is at the bottom of the potential well. Hence

$$J=\oint\sqrt{[2E-2U(f)]}df=\sqrt{\frac{d^2U}{df^2}|_{f^*}}\oint\sqrt{a^2-\delta f^2}d\,\delta f,$$

where $a^2=2\,\delta E/[(d^2U/df^2)|_{f^*}]$. This integral is evaluated directly to give

$$J=\frac{2\pi\delta E}{\sqrt{\left[\frac{d^2U}{df^2}\right]_{f^*}}}.$$

Since the phase-space orbit is a circle, $\sqrt{2}(\delta E)^{1/2}/[d^2U/df^2|_{f^*}]^{1/4}$ will be its radius. When $\delta E=0$, $J=0$.

Also $R=1/[(d^2U/df^2)|_{f^*}]$ is the radius of curvature ($R\gg1$) of the potential curve $U(f)$ vs f near f^*. Then, $R^2-(R-\Delta E)^2=(\delta f)^2$ or $2R\Delta E\simeq(\delta f)^2$ where $|\Delta E|\ll R$ implies that $\Delta E/[(d^2U/df^2)|_{f^*}]\simeq(\delta f)^2/2=J/2\pi$ so that $J=\pi(\delta f)^2$. This form of J is also obvious from the fact that the phase-space orbit may be considered to be a circle of radius $\delta f=f-f^*$.

This evaluation of the radial action variable implies that the radial (spatial) period is given through [using Eq. (52)]

$$\nu=\frac{1}{2\pi}\sqrt{\left[\frac{d^2U}{df^2}\right]_{f^*}}=\frac{1}{2\eta_f}. \tag{55}$$

The solid straight line independent of E in Fig. 8 is the depiction of z_f from this expression at the bottom of the potential well.

The action J in the above analysis is actually the r component of the action, $J_r=J_f$. There will be a corresponding θ component, $J_{\theta_f}=\oint p_{\theta_f}d\theta_f$ which goes on to define another (θ_f) periodicity through $\nu_{\theta_f}=dE/dJ_{\theta_f}$. In general, the radial frequency ν and the angle frequency ν_{θ_f} are incommensurate, i.e., $\nu/\nu_{\theta_f}\neq$ integer, implying that the orbit in the (f, θ_f) plane is ergodic in the annular region between the minimum and the maximum values of f viz., f_m and f_M, respectively. This behavior is seen in Fig. 6 with the orbit filling up the annular region with increasing values of η.

IV. SELF-FOCUSING IN AN ABSORBING MEDIUM

In an absorbing medium, the formulations of Sec. II and the preceding section are not suitable. The general advantage that in principle existed in Sec. II that the beam need not be of Gaussian nature at $z=0$ and that Eq. (11) can in principle be directly dealt with, will fully be given up in this section and a theory valid only for a Gaussian beam will be adopted. This is a compromise necessary at present to build up a self-focusing theory for an absorbing medium.

The quasioptic equation, Eq. (2), is again the starting point but with the value of the propagation constant k given by $k^2=k_0^2\epsilon_{Lr0}$ and $\epsilon_{Lr0}=1-\omega_p^2/\omega^2$. We go on to write the solution of the scalar wave equation in terms of the Gaussian self-similar form

$$E(r,z)=E_0e^{-i\phi(z)}e^{-r^2/2r_0^2g(z)}, \tag{56}$$

where $g(z)$ is the complex beamwidth parameter and $\phi(z)$ is the on-axis phase shift. Substituting this field form into the quasioptic scalar wave equation and using the dielectric field expansion of Eq. (15) which is still valid if $1/f^2$ is suitably replaced by its equivalent given in Sec. IV B below [cf. Eq. (65)], we get two equations as coefficients of the r^2-independent terms and r^2-dependent terms. The two equations are, respectively,

$$\phi(z)=\frac{-1}{kr_0^2}\int_0^z\frac{dz}{g}+\frac{k_0}{2\sqrt{\epsilon_L}}\int_0^z(\epsilon_0-\epsilon_L)dz \tag{57}$$

for the on-axis phase and the complex beamwidth, g,

$$\frac{dg}{dz}=\frac{i}{r_0^2k}[r_0^4g^2\epsilon_2k_0^2-1]. \tag{58}$$

A. The *ABCD* law

With $\eta=z/kr_0^2$ we get from Eq. (58)

$$\frac{d}{d\eta}\left(\frac{1}{g}\right)=-i\left[r_0^4\epsilon_2k_0^2-\frac{1}{g^2}\right].$$

Let $g=-i\zeta/\zeta'$ where $\zeta'=d\zeta/d\eta$. This reduces the above equation to

$$\frac{d^2\zeta}{d\eta^2}=-r_0^4\epsilon_2k_0^2\zeta. \tag{59}$$

Solving the Ricatti-like equation for g, Eq. (58), or solving this simple-harmonic motion equation (with η-dependent frequency term) in ζ are equivalent. For small $\Delta\eta$, one regards $\epsilon_2(g_r,g_i)$ as a constant and obtains a simple solution. First, we assume the following approximate solution:

$$\zeta(\eta)\cong A\sin\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right]+B\cos\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right]. \tag{60}$$

At $\eta=0$, $g=g_0$ and we have $B/A=i\widetilde{g_0}=ig_0k_0\sqrt{\epsilon_2}r_0^2$. We define $\widetilde{g}=r_0^2\sqrt{\epsilon_2}gk_0$ so that we can now write using $g=-i\zeta/\zeta'$,

$$\widetilde{g}=\frac{\widetilde{g_0}\cos\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right]-i\sin\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right]}{-i\widetilde{g_0}\sin\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right]+\cos\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right]} \tag{61}$$

So the *ABCD* law for $\widetilde{g}$ involves the matrix,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

given by

$$\begin{pmatrix} \cos\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right] & -i\sin\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right] \\ -i\sin\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right] & \cos\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right] \end{pmatrix}. \tag{62}$$

This has the advantage that it can repeatedly be applied for each optical element and the final *ABCD* matrix obtained by multiplication serves to get the final value of $\widetilde{g}$ by substitution into the bilinear expression, Eq. (61) [33].

Note that the above matrix can be written as

$$Q=\sigma_0\cos\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right]-i\sigma_1\sin\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right], \tag{63}$$

where $\sigma_0=\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $\sigma_1=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Q can also be written as

$$Q=\exp\left(-i\sigma_1\left[k_0r_0^2\int_0^\eta\sqrt{\epsilon_2}d\eta\right]\right), \tag{64}$$

so that the needed *ABCD* matrix is actually a rotation matrix at least as long as ϵ_2 is real. When ϵ_2 is complex, this becomes the Lorentz transformation matrix. We can deal with the self-focusing medium as a piecewise linear medium, each piece $\Delta\eta$ in thickness. The local value of $\sqrt{\epsilon_2}$ is used for each of these slabs of thickness $\Delta\eta$. Also the Q's commute with each other so we just add exponents since σ_1 commutes with itself. ($e^Ae^B=e^{A+B}$ if $[A,B]=0$.) Using this we can have a fast and useful algorithm for self-focusing calculations.

B. Connecting with the absorptionless case

At least when there is no absorption, it is convenient to switch over to the $(f,df/d\eta)$ variables. We have the equivalency

$$\frac{1}{g}=\frac{1}{f^2}+i\frac{1}{f}\frac{df}{d\eta} \tag{65}$$

$$\Rightarrow\frac{1}{f^2}=\frac{g_r}{g_r^2+g_i^2},\quad \frac{1}{f}\frac{df}{d\eta}=-\frac{g_i}{g_r^2+g_i^2},$$

where $\eta=z/kr_0^2$. Using these equivalencies in Eq. (58), we immediately get Eq. (19) for f which should work well if ϵ_2 is real since we need f to be always real.

Note also that $\phi=\phi_1+\phi_2$,

$$\phi_1=\frac{-1}{kr_0^2}\int_0^z\frac{dz}{g}=-\int^\eta\frac{d\eta}{g}=-\int^\eta\frac{d\eta}{f^2}-i\ln f, \tag{66}$$

where the first term on the right is the Gouy phase and

$$\phi_2=\frac{k_0}{2\epsilon_L}\int^z(\epsilon_0-\epsilon_L)dz \tag{67}$$

is the dynamical phase. The field assumes the form

$$E=E_0e^{-i\phi(z)}e^{-r^2/2r_0^2g(z)}=E_0e^{-i\phi_1}e^{-i\phi_2}e^{-r^2/2r_0^2g(z)} \tag{68}$$

or

$$E(r,\eta)=E_0\frac{1}{f}\exp\left(i\int^\eta\frac{d\eta}{f^2}\right)e^{-i\phi_2}\exp\left(\frac{-r^2}{2r_0^2f^2}\right)\times\exp\left(\frac{-ir^2}{2r_0^2}\frac{1}{f}\frac{df}{d\eta}\right) \tag{69}$$

so that the intensity factor is

$$|E(r,\eta)|^2=\frac{E_0^2}{f^2}\exp\left(\frac{-2r^2}{2r_0^2f^2}\right)=\frac{E_0^2}{f^2}\exp\left(\frac{-r^2}{r_0^2f^2}\right). \tag{70}$$

Equation (69) shows that the phase in Eq. (20) is now recovered whenever ϕ_2 is real. (If ϕ_2 is complex, $e^{-i\phi_2}=e^{-i\phi_{2r}}e^{\phi_{2i}}$ and

$$|E(r,\eta)|^2=\frac{E_0^2}{f^2}\exp\left(\frac{-r^2}{r_0^2f^2}\right)e^{2\phi_{2i}}. \tag{71}$$

The last term in ϕ_{2i} should have the correct sign in the exponent to show decay in z in an absorbing medium giving the nonlinear Lambert's law.)

We also sometimes need the relations for g_r and g_i in terms of f and its derivative which can easily be worked out to be

$$g_r=\frac{\frac{1}{f^2}}{\frac{1}{f^4}+\left(\frac{1}{f}\frac{df}{d\eta}\right)^2},\quad g_i=\frac{-\frac{1}{f}\frac{df}{d\eta}}{\frac{1}{f^4}+\left(\frac{1}{f}\frac{df}{d\eta}\right)^2}. \tag{72}$$

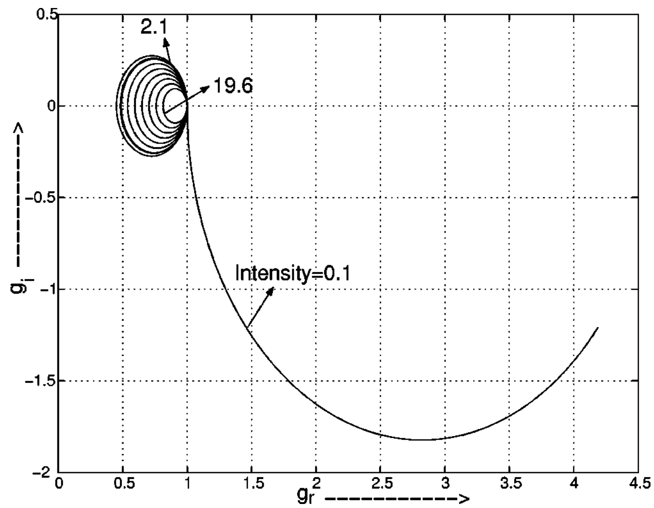

FIG. 9. Phase-space orbits in the complex g-plane for self-focusing corresponding to the plots of real beamwidth parameter in Fig. 5 for the absorptionless case $\omega_p r_0 / c = 3.0$.

C. Self-focusing dynamics in an absorbing medium

The complex beamwidth equation, Eq. (58), along with the nonlinear Lambert's law in Eq. (71) (with the f terms replaced by the equivalent terms in g_r and g_i) has been numerically solved to obtain the orbit of the focusing beam on the complex g plane. The expression for ϵ_2 from Sec. II C for ponderomotive nonlinearity is used with (ω_p^2/ω^2) replaced suitably by $\omega_p^2/\omega^2)(1 - i\nu/\omega)^{-1}$ for the absorbing plasma, where ν is the plasma collision frequency. Figure 9 gives first the orbits in the complex g-plane in the absence of absorption that are equivalent to Fig. 5 for the phase plots of f obtained now with the help of Eq. (72). Figure 10 gives for the same parameters the complex g plane orbit when absorption is present. The result that the center point in phase space of the absorptionless case that gave a closed orbit is now converted to a focus in the presence of absorption is a very significant result that is important for all realistic media such as an absorbing plasma.

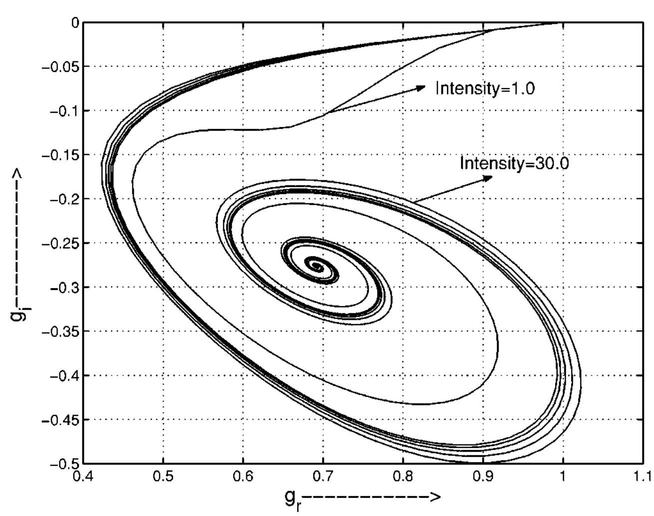

FIG. 10. Self-focusing phase plot in complex g-plane with absorption $\omega_p r_0 / c = 3.0$, $\nu/\omega = 0.4$. Notice the convergence to the fixed point dictated by Eq. (73) suggesting beam self-organization.

The focus itself is situated at the fixed point of the complex beamwidth equation, Eq. (58). Equating the right hand side to zero gives the fixed point as

$$g^* = \frac{1}{k_0 r_0^2 \epsilon_2^{1/2}}, \tag{73}$$

which is now complex when ϵ_2 is complex. Since ϵ_2 is nonlinear and involves g and $\overline{g}$, this complex equation is equivalent to two simultaneous nonlinear equations to be solved together. One manner of solving them is by iteration. The iteration can be done on any path on the complex g-plane that terminates at the fixed point and has some suitable initial guess on the g-plane, provided there is no irreversibility on the complex g-plane introduced by the nonanalyticity of the functions involved (e.g., ϵ_2 is a function of both g and $\overline{g}$ that makes it nonanalytic in g). Of all the possible paths we chose for iteration the actual beamwidth development path on the g-plane by solving the complex beamwidth equation, Eq. (58) and verifying that as $\eta \to \infty$ the iterates of g are such that both sides of the equation vanish. That ensures that the fixed point has been reached through iteration as in Fig. 10 along the actual path followed by the complex beamwidth. The location of the fixed point and hence the values of (g_r^*, g_i^*) depend on the extent of absorption, i.e., on the value of ν/ω. This in turn implies a certain beamwidth and a constant small phase-front curvature to which the beam evolves for sufficiently large η (~ 160).

V. CONCLUSIONS AND DISCUSSION

The self-focusing beamwidth equation, Eq. (19), and that for the phase evolution, Eq. (20), are valid in general for all absorptionless saturating nonlinearities. The specialization to plasma nonlinearities has shown that these equations can be exploited to study self-focusing in an absorptionless medium using the potential function in Sec. III B. The similarity of the problem of self-focusing in an absorptionless medium to the problem of dynamics in a central potential in Sec. III C is too close to ignore. In particular, the identification of the problem of diffraction to the conservation of angular momentum in the (f, θ_f) space in Sec. III C remains a curious result. This result is reminiscent of a similar result in the context of a more direct development of the problem of the time- or η-dependent harmonic oscillator in the classic work of Lewis and Riesenfeld [34]. Note that with a potential or rather the dielectric constant of the form of Eq. (15), the present theory also deals with a time- or η-dependent harmonic oscillator problem although our aim had been to extract only the information on the field envelope and phase unlike the direct calculation of the eigenfunctions as in Lewis and Riesenfeld [34]. To our knowledge, the possibilities of self-trapping of the second kind pointed out in Sec. III B is new. The method of evaluating the period of self-focusing oscillations in Sec III C using action variable should prove useful.

The practical problem of self-focusing in an absorbing medium needed a complete reformulation of the self-focusing problem in Sec. IV in terms of the complex beamwidth parameter that reduced to the earlier absorptionless case involving only the real beamwidth parameter f and its derivative in Sec. IV B. The results of the section obtained by numerical solution of the complex beamwidth equation in Sec. IV C in an absorbing medium are indeed very impressive in the sense that they always indicate the convergence of the beam to a focus in phase space. This existence of an attractor point in phase space indicates that the complex beamwidth converges to some constant value after propagation through several Rayleigh lengths and is qualitatively supportive of the experimental result on damage tracks left in self-focusing media that are seen to have constant radius for long distances [35]. The beam phase front need not be plane but has a small curvature as witnessed from the fact that $g_i^* \neq 0$ for the attractor. The presence of the attractor in phase space is also indicative of the general self-organization of the self-focusing beam.

The result of the absorbing medium is interesting also from the point of view that any effective absorption that gets introduced in terms of the radiation of the beam will imply asymptotically to constant beamwidth situations indicative of self-organization of the self-focusing beam in an absorptionless medium. The estimation of radiation by not using the gaussian ansatz for the beam profile but the full beam equation that results in Sec. II [Eq. (11)] then should give a quantitative theory that indicates whether or not such a focusing to a constant radius situation occurs.

To elaborate this further, we note that with the notation $\tilde{A}_0 = F^{1/2}\xi^{1/2}$, Eq. (11) can be put into the form

$$\frac{d^2\tilde{A}_0}{d\xi^2} + \left[a^2\xi^2 - b^2 + \frac{1}{4\xi^2} \right] \tilde{A}_0 = 0, \tag{74}$$

where

$$a^2 = f^2 k_0^2 r_0^2 S_2\left(\frac{E_0^2}{f^2}\right) - f^3 \frac{d^2 f}{d\eta^2},$$

$$b^2 = 2f^2 \frac{d\phi}{d\eta} + f^2 k_0^2 r_0^2 (\epsilon_L - \epsilon_0).$$

The bound solutions of Eq. (74) are known in the form of the Laguerre' Gaussians, $\tilde{A}_0 = \exp(-\xi^2/2)\xi^{1/2}L_m(\xi^2)$ provided we choose $a^2 = -1$ and $b^2 = -(4m+2)$ in Eq. (74) (cf. Ref. [31], p. 781, Table 22.6). The choice $m=0$ then gives the Gaussian solution given in the paper with the beamwidth equation and axial phase equations given by Eqs. (19) and (20), respectively. The higher-order m solutions can go on to give the self-focusing description of the higher-order laser modes if they are introduced into the nonlinear plasma, provided the on-axis beam phase equation is taken according to the condition $b^2 = -(4m+2)$. This topic will not be elaborated here further. All these solutions including the Gaussian solution of this paper, however, could be rendered unstable owing to the fact that Eq. (11) demands that as $\xi \to \infty$, $\Phi \to 0$ since $F^{1/2} \to 0$ there and $\Phi(0)=0$ by definition from Eqs. (3) and (4). This in turn demands that Eq. (74) be modified to a similar equation with a new definition of $a^2 = -f^3(d^2f/d\eta^2)$ in the region $\xi \to \infty$. This, along with the condition $a^2 = -1$ and $b^2 = -(4m+2)$ necessary for bound solutions given earlier, gives a beamwidth equation in the region $\xi \to \infty$ that simply indicates laser beam diffraction. Combining also with the fact that the beam wave front dictated by the eikonal of Eq. (7) also becomes plane as $r,\xi \to \infty$ as the curvature formula indicates [that is given in the equation after Eq. (7)], the beam tends to be diffracting plane wave front in this region. This leads to energy leakage from the central portion of the beam that is Gaussian (or Laguerre' Gaussian in nature) and hence contributes to an instability that needs to be investigated in future. The reorganization of the Gaussian beam is expected because of these developments starting from the description of the beam in the early stages as in this paper. Energy leakage from the central part of the beam is expected to contribute an imaginary term into the square bracket in Eq. (74). As pointed out in this paper (Sec. IV), this will be similar to adding an effective absorption to the central part of the beam that should stabilize the laser beam towards the bottom of the potential well in any case. This needs detailed analysis in future.

The present method also gives, in principle, a way to reduce the wave equation to a large number of coupled differential equations, more than two unlike this paper, using higher power expansions of the dielectric constant than Eq. (15) that can be explored in future. That should be another method to get rid of the self-similar propagation of the beam and study the details of the reorganization of the beam during propagation.

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[27] As a computer algorithm it is useful sometimes to first calculate S_2 directly using the integral expression,

$$S_2(x)=\frac{1}{x}\int_0^x\left(\frac{1}{y}\int_0^y dz z^2\,\frac{d\Phi}{dz}\right)dy$$

or

$$S_2(x)=\frac{1}{x}\int_0^x\left(y\Phi(y)-\frac{2}{y}I(y)\right)dy,$$

where $I(y)=\int_0^y z\Phi(z)dz$, so that two ordinary differential equations

$$\frac{dI}{dy}=y\Phi(y),\ \frac{d\overline{Q}}{dy}=\frac{dI}{dy}-\frac{2}{y}I(y)$$

with $\overline{Q}=S_2 y$ completely solve for S_2. One can then solve for $S(x)$ using $S_2(x)=x(d/dx)S(x)$ or $S(x)=\int_0^x(1/x)S_2(x)dx$. Lower limit 0 is decided by the fact that $S(x)\to 0$ as $x\to 0$ and $S_2(x)\to 0$ as $x\to 0$. Numerical integration can give $S(x)$.

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